Ejercicios Teoría Cuántica de Campos. Capítulo 31

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1 Calcular $e^{\frac{\theta}{2}m_0m_1}m_1e^{-\frac{\theta}{2}m_0m_1}$.

Vamos a calcular el caso general de

$$e^{\frac{\theta}{2}m_0m_1}m_\alpha e^{-\frac{\theta}{2}m_0m_1}, \qquad \alpha = 0, 1 \tag{1}$$

Sabemos que

$$e^{\frac{\theta}{2}m_0m_1} = \cosh\frac{\theta}{2} + m_0m_1\sinh\frac{\theta}{2}, \qquad e^{-\frac{\theta}{2}m_0m_1} = \cosh\frac{\theta}{2} - m_0m_1\sinh\frac{\theta}{2} \tag{2}$$

Por lo tanto, sustituyendo estas ecuaciones

$$e^{\frac{\theta}{2}m_0m_1}m_\alpha e^{-\frac{\theta}{2}m_0m_1} = \left(\cosh\frac{\theta}{2} + m_0m_1\sinh\frac{\theta}{2}\right)m_\alpha\left(\cosh\frac{\theta}{2} - m_0m_1\sinh\frac{\theta}{2}\right)$$

Usando la propiedad distributiva dos veces obtenemos

$$\begin{aligned} e^{\frac{\theta}{2}m_0m_1}m_\alpha e^{-\frac{\theta}{2}m_0m_1} = \cosh\frac{\theta}{2}m_\alpha\cosh\frac{\theta}{2} - \cosh\frac{\theta}{2}m_\alpha m_0m_1\sinh\frac{\theta}{2} + m_0m_1\sinh\frac{\theta}{2}m_\alpha\cosh\frac{\theta}{2} \\ - m_0m_1\sinh\frac{\theta}{2}m_\alpha m_0m_1\sinh\frac{\theta}{2} \end{aligned}$$

Como θ es un número y por lo tanto conmuta, y usando la propiedad de las funciones hiperbólicas

$$2\cosh\alpha\sinh\alpha = \sinh 2\alpha \tag{3}$$

$$\begin{aligned} e^{\frac{\theta}{2}m_0m_1}m_\alpha e^{-\frac{\theta}{2}m_0m_1} &= \cosh^2\frac{\theta}{2}m_\alpha - \sinh\frac{\theta}{2}\cosh\frac{\theta}{2}\left(m_\alpha m_0m_1 - m_0m_1m_\alpha\right) - \sinh^2\frac{\theta}{2}m_0m_1m_\alpha m_0m_1 \\ &= \cosh^2\frac{\theta}{2}m_\alpha - \frac{\sinh\theta}{2}\left(m_\alpha m_0m_1 - m_0m_1m_\alpha\right) - \sinh^2\frac{\theta}{2}m_0m_1m_\alpha m_0m_1 \end{aligned}$$

Usando las propiedades $m_\alpha m_\beta = 2g_{\alpha\beta} - m_\beta m_\alpha$ podemos simplificar los productos de matrices:

$$\begin{aligned} m_\alpha m_0m_1 - m_0m_1m_\alpha &= \left(2g_{\alpha0} - m_0m_\alpha\right)m_1 - m_0\left(2g_{1\alpha} - m_\alpha m_1\right) \\ &= 2g_{\alpha0}m_1 - m_0m_\alpha m_1 - 2g_{1\alpha}m_0 + m_0m_\alpha m_1 \\ &= 2\left(g_{\alpha0}m_1 - g_{\alpha1}m_0\right) \end{aligned}$$

$$\begin{aligned} m_0m_1m_\alpha m_0m_1 &= m_0\left(2g_{1\alpha} - m_\alpha m_1\right)m_0m_1 = 2g_{1\alpha}m_0m_0m_1 - m_0m_\alpha m_1m_0m_1 = 2g_{1\alpha}m_1 - m_0m_\alpha m_0 \\ &= 2g_{1\alpha}m_1 - m_0\left(2g_{\alpha0} - m_0m_\alpha\right) = 2g_{1\alpha}m_1 - 2g_{\alpha0}m_0 + m_0m_0m_\alpha \\ &= 2g_{\alpha1}m_1 - 2g_{\alpha0}m_0 + m_\alpha \end{aligned}$$

Esta expresión se puede simplificar usando deltas de Kronecker; por ejemplo, fijaos que podemos escribir $g_{\alpha0} = \delta_{\alpha0}$ y $g_{\alpha1} = -\delta_{\alpha1}$, quedando más simplificada

$$m_0m_1m_\alpha m_0m_1 = -2\delta_{\alpha1}m_1 - 2\delta_{\alpha0}m_0 + m_\alpha = -2\delta_{\alpha\beta}m_\beta + m_\alpha = -m_\alpha \tag{4}$$

Donde he usado el criterio de sumación de Einstein.

Sustituyendo estas expresiones en la transformación de m_α obtenemos

$$e^{\frac{\theta}{2}m_0 m_1} m_\alpha e^{-\frac{\theta}{2}m_0 m_1} = \cosh\theta m_\alpha - \sinh\theta\left(g_{\alpha 0} m_1 - g_{\alpha 1} m_0\right) = \cosh\theta m_\alpha - \sinh\theta\left(\delta_{\alpha 0} m_1 + \delta_{\alpha 1} m_0\right)$$

Que, primero de todo, concuerda con la expresión calculada por Javier

$$e^{\frac{\theta}{2}m_0 m_1} m_0 e^{-\frac{\theta}{2}m_0 m_1} = \cosh\theta m_0 - \sinh\theta\left(\delta_{00} m_1 + \delta_{01} m_0\right) = \cosh\theta m_0 - \sinh\theta m_1$$

Y, además generaliza el resultado a

$$e^{\frac{\theta}{2}m_0 m_1} m_1 e^{-\frac{\theta}{2}m_0 m_1} = \cosh\theta m_1 - \sinh\theta\left(\delta_{10} m_1 + \delta_{11} m_0\right) = \cosh\theta m_1 - \sinh\theta m_0$$

En resumen, podemos escribir la transformación de m como

$$e^{\frac{\theta}{2}m_0 m_1} m_\alpha e^{-\frac{\theta}{2}m_0 m_1} = \Lambda_{\alpha\beta} m_\beta \tag{5}$$

Con

$$\Lambda_{\alpha\beta} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \tag{6}$$